Periodic signals

Guillaume Frèche

Version 1.0

Definition 0.1 (Periodic signal, fundamental period)
A signal $x \in \mathcal{F}(\mathbb{R}, \mathbb{K})$ is **periodic** if there exists $T > 0$ such that for any $t \in \mathbb{R}$, $x(t+T) = x(t)$. The smallest $T_0 > 0$ such that $x(t+T_0) = x(t)$ for any $t \in \mathbb{R}$ is the **fundamental period**. We denote $\mathcal{F}_{T_0}(\mathbb{R}, \mathbb{K})$ the subspace of periodic signals with period T_0.

Remarks:

- If a signal x is periodic with period T_0, then it is with period kT_0 for any $k \in \mathbb{Z}$. This is a cornerstone property intensively used in the next part.
- In other words, a signal x is periodic with period T if it is invariant by a pure delay of T, i.e. $\tau_T(x) = x$.

Definition 0.2 (Fundamental frequency, fundamental impulse)
Let $x \in \mathcal{F}_{T_0}(\mathbb{R}, \mathbb{K})$ be a periodic signal with period T_0. The **fundamental frequency** of x is the number $f_0 = \dfrac{1}{T_0}$, and the **fundamental impulse** of x is the number $\omega_0 = 2\pi f_0 = \dfrac{2\pi}{T_0}$.

Definition 0.3 (Complex exponential, cosine)
The **complex exponential** of fundamental impulse ω_0, amplitude $A > 0$ and phase $\varphi_0 \in [0, 2\pi[$ is the following signal in $\mathcal{F}_{T_0}(\mathbb{R}, \mathbb{C})$:

$$e_{\omega_0, A, \varphi_0} : t \mapsto A \exp\left(i(\omega_0 t + \varphi_0)\right)$$

We simply denote $e_{\omega_0} = e_{\omega_0, 1, 0}$ the complex exponential of amplitude $A = 1$ and phase $\varphi_0 = 0$.
The **cosine** of fundamental impulse ω_0, amplitude $A > 0$ and phase $\varphi_0 \in [0, 2\pi[$ is the following signal in $\mathcal{F}_{T_0}(\mathbb{R}, \mathbb{C})$:

$$c_{\omega_0, A, \varphi_0} : t \mapsto A \cos(\omega_0 t + \varphi_0)$$

We simply denote $c_{\omega_0} = c_{\omega_0, 1, 0}$ the cosine of amplitude $A = 1$ and phase $\varphi_0 = 0$.

Remarks:

- With these definitions, we can check that the complex exponential and cosine of fundamental impulse ω_0 are periodic signals with period $T_0 = \dfrac{2\pi}{\omega_0}$.
- If a periodic signal with period T_0 is the input of an LTI system, then the corresponding output is with period T_0 as well. Indeed let an LTI system L, a periodic signal x with period T_0, et $y = L(x)$ the corresponding output. Since LTI systems commute with pure delays,

  $$\tau_{T_0}(y) = \tau_{T_0}(L(x)) = L(\tau_{T_0}(x)) = L(x) = y$$

  thus y is also periodic with period T_0.

- As we are going to see in Example 0.1, we often deal in practice with signals of the form $t \mapsto e_{\omega_0,A,\varphi_0}(t)\Upsilon(t)$ and $t \mapsto c_{\omega_0,A,\varphi_0}(t)\Upsilon(t)$, namely zero over $]-\infty, 0[$ and oscillating over $[0, +\infty[$. These signals will be the matter of a future lecture.

If a signal x is periodic with period T_0, signal $t \mapsto |x(t)|^2$ is clearly periodic with period T_0 as well. Thus for any $n \in \mathbb{N}^*$,

$$\int_{-nT_0}^{nT_0} |x(t)|^2 dt = 2n \int_0^{T_0} |x(t)|^2 dt$$

When n goes to $+\infty$, we note that a non-zero periodic signal has infinite energy. However, the average power of such a periodic signal x is:

$$P(x) = \lim_{t\to+\infty} \frac{1}{2t} \int_{-t}^{t} |x(u)|^2 du = \lim_{n\to+\infty} \frac{1}{2nT_0} \int_{-nT_0}^{nT_0} |x(u)|^2 du = \frac{1}{T_0} \int_0^{T_0} |x(u)|^2 du$$

We are going to define a subspace of $\mathcal{F}_{T_0}(\mathbb{R}, \mathbb{K})$ containing periodic signals with period T_0 which are locally square integrable, i.e. they have a finite average power.

Lemma 0.1
Let x be a periodic signal with period T_0. For any $a \in \mathbb{R}$,

$$\int_a^{a+T_0} x(t)dt = \int_0^{T_0} x(t)dt$$

Proof : If $a \in [0, T_0]$, then $T_0 \in [a, a + T_0]$. By the change of variable $t \mapsto t - T_0$, we get

$$\int_a^{a+T_0} x(t)dt = \int_a^{T_0} x(t)dt + \int_{T_0}^{a+T_0} x(t)dt = \int_a^{T_0} x(t)dt + \int_0^{a} x(t+T_0)dt = \int_0^{T_0} x(t)dt$$

In general, let $a \in \mathbb{R}$. If $b = a - \left\lfloor \frac{a}{T_0} \right\rfloor T_0$, then $b \in [0, T_0]$ and by the change of variable $t \mapsto t - \left\lfloor \frac{a}{T_0} \right\rfloor T_0$,

$$\int_a^{a+T_0} x(t)dt = \int_b^{b+T_0} x(t)dt = \int_0^{T_0} x(t)dt \qquad \blacksquare$$

This lemma indicates that the integral of a periodic signal with period T_0 is identical on any interval of length T_0. Therefore, we can now define the subspace of signals with finite average power and define on this subspace a scalar product base on the average power, instead of the energy which is infinite.

Definition 0.4
We denote $L^2_{T_0}(\mathbb{R}, \mathbb{K})$ the subspace of $\mathcal{F}_{T_0}(\mathbb{R}, \mathbb{K})$ containing the periodic signals with period T_0 which are square integrable over $[0, T_0]$, i.e.

$$L^2_{T_0}(\mathbb{R}, \mathbb{K}) = \left\{ x \in \mathcal{F}_{T_0}(\mathbb{R}, \mathbb{K}), \frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt < +\infty \right\}$$

Definition 0.5
We define a **scalar product / Hermitian product** over $L^2_{T_0}(\mathbb{R}, \mathbb{K})$ by

$$\forall (x, y) \in L^2_{T_0}(\mathbb{R}, \mathbb{K})^2 \qquad \langle x, y \rangle_{T_0} = \frac{1}{T_0} \int_0^{T_0} x(t)y^*(t)dt$$

From this scalar / Hermitian product, we can define the norm of a signal x to which we can connect the average power of the signal:

$$\forall x \in L^2_{T_0}(\mathbb{R}, \mathbb{K}) \qquad P(x) = \|x\|^2_{T_0} = \langle x, x \rangle_{T_0}$$

i.e.

$$\forall x \in L^2_{T_0}(\mathbb{R}, \mathbb{K}) \quad P(x) = \frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt$$

Remarks:

- We defined these integrals over the interval $[0, T_0]$ but according to the lemma, any interval of length T_0 is suitable. In some cases, it is more interesting to exploit the symmetry of interval $\left[-\frac{T_0}{2}, \frac{T_0}{2}\right]$, when we deal with odd or even signals for example.
- We can define cross-correlation and autocorrelation of periodic signals from this new scalar product.

Proposition 0.2
The autocorrelation of a periodic signal with period T_0 is also a periodic signal with period T_0.

PROOF : Let $x \in L^2_{T_0}(\mathbb{R}, \mathbb{K})$.

$$\forall t \in \mathbb{R} \qquad \gamma_x(t + T_0) = \langle x, \tau_{t+T_0}(x) \rangle = \langle x, \tau_t(x) \rangle = \gamma_x(t)$$

because the periodicity of x implies $\tau_{t+T_0}(x) = \tau_t\left(\tau_{T_0}(x)\right) = \tau_t(x)$. ■

Now we study the convolution of two non-zero periodic signals x and y with the same period T_0. Let $t \in \mathbb{R}$. Then signals $u \mapsto y(t-u)$ and $u \mapsto x(u)y(t-u)$ are also periodic with period T_0, thus

$$(x * y)(t) = \int_{-\infty}^{+\infty} x(u)y(t-u)du = \lim_{n \to +\infty} \int_{-nT_0}^{nT_0} x(u)y(t-u)du = \lim_{n \to +\infty} 2n \int_0^{T_0} x(u)y(t-u)du = +\infty$$

This result is not surprising, since the notions of energy and convolution are connected through correlation, and non-zero periodic signals have infinite energy. As for the scalar product, we have to adapt our definition of convolution.

Definition 0.6 (Circular convolution)
The **circular convolution** $\otimes$ is a product in $\mathcal{F}_{T_0}(\mathbb{R}, \mathbb{K})$ defined by

$$\forall (x, y) \in \mathcal{F}_{T_0}(\mathbb{R}, \mathbb{K})^2 \qquad \forall t \in \mathbb{R} \qquad (x \otimes y)(t) = \frac{1}{T_0} \int_0^{T_0} x(u)y(t-u)du$$

Remark: The circular convolution of two periodic signals x and y with period T_0 is also periodic with period T_0. Indeed, for any $t \in \mathbb{R}$,

$$(x \otimes y)(t + T_0) = \frac{1}{T_0} \int_0^{T_0} x(u)y(t + T_0 - u)du = \frac{1}{T_0} \int_0^{T_0} x(u)y(t-u)du = (x \otimes y)(t)$$

Example 0.1

We go back to the RC circuit and we look for its response to the input $V(t) = \sin(\omega_0 t)\Upsilon(t) = \cos\left(\omega_0 t - \frac{\pi}{2}\right)\Upsilon(t)$, which is periodic over $[0, +\infty[$. We determine this response with two techniques developped so far: solving the governing differential equation and computing the convolution with the impulse response.

To abbreviate computations, we set $\tau = RC$, the time constant of the circuit. Recall that the solutions of the homogeneous differential equation are of the form $u_c(t) = K\exp\left(-\frac{t}{\tau}\right)$, with $K \in \mathbb{R}$. We look for a particular solution of the form $u_c(t) = A\sin(\omega_0 t) + B\cos(\omega_0 t)$ over $[0, +\infty[$. The derivative of such a function is $u_c'(t) = A\omega_0\cos(\omega_0 t) - B\omega_0\sin(\omega_0 t)$. Then the differential equation becomes over $[0, +\infty[$:

$$(A - B\tau\omega_0)\sin(\omega_0 t) + (A\tau\omega_0 + B)\cos(\omega_0 t) = \sin(\omega_0 t)$$

By identification, we get $A - B\tau\omega_0 = 1$ and $A\tau\omega_0 + B = 0$, yielding

$$A = \frac{1}{1+\tau^2\omega_0^2} \qquad \text{and} \qquad B = -\frac{\tau\omega_0}{1+\tau^2\omega_0^2}$$

Thus we have the solution

$$u_c(t) = \begin{cases} 0 & \text{if } t < 0 \\ K\exp\left(-\frac{t}{\tau}\right) + \frac{1}{1+\tau^2\omega_0^2}\sin(\omega_0 t) - \frac{\tau\omega_0}{1+\tau^2\omega_0^2}\cos(\omega_0 t) & \text{if } t > 0 \end{cases}$$

Since u_c in continuous in $t = 0$,

$$\lim_{t\to 0^-} u_c(t) = 0 = \lim_{t\to 0^+} u_c(t) = K - \frac{\tau\omega_0}{1+\tau^2\omega_0^2}$$

for any $t \in [0, +\infty[$,

$$u_c(t) = \frac{\tau\omega_0}{1+\tau^2\omega_0^2}\exp\left(-\frac{t}{\tau}\right) + \frac{1}{1+\tau^2\omega_0^2}\sin(\omega_0 t) - \frac{\tau\omega_0}{1+\tau^2\omega_0^2}\cos(\omega_0 t)$$

Now we want to retrieve this result by the convolution of $V(t)$ with impulse response $h(t) = \frac{1}{\tau}\exp\left(-\frac{t}{\tau}\right)\Upsilon(t)$. For $t \in]-\infty, 0[$, the supports of $u_c(u)$ and $h(t-u)$ are disjoint, thus $u_c(t) = (V * h)(t) = 0$. For $t \in [0, +\infty[$,

$$\begin{aligned} u_c(t) = (V * h)(t) &= \int_{-\infty}^{+\infty} \Upsilon(u)\sin(\omega_0 u)\frac{1}{\tau}\exp\left(-\frac{t-u}{\tau}\right)\Upsilon(t-u)du \\ &= \frac{1}{\tau}\exp\left(-\frac{t}{\tau}\right)\int_0^t \sin(\omega_0 u)\exp\left(\frac{u}{\tau}\right)du \end{aligned}$$

A double integration by parts (left to the interested reader) gives

$$\int_0^t \sin(\omega_0 u)\exp\left(\frac{u}{\tau}\right)du = \frac{\tau^2\omega_0}{1+\tau^2\omega_0^2} + \frac{\tau}{1+\tau^2\omega_0^2}\sin(\omega_0 t)\exp\left(\frac{t}{\tau}\right) - \frac{\tau^2\omega_0}{1+\tau^2\omega_0^2}\cos(\omega_0 t)\exp\left(\frac{t}{\tau}\right)$$

Hence we retrieve:

$$u_c(t) = \frac{\tau\omega_0}{1+\tau^2\omega_0^2}\exp\left(-\frac{t}{\tau}\right) + \frac{1}{1+\tau^2\omega_0^2}\sin(\omega_0 t) - \frac{\tau\omega_0}{1+\tau^2\omega_0^2}\cos(\omega_0 t)$$

Note that $\lim_{t \to +\infty} \exp\left(-\frac{t}{\tau}\right) = 0$ so that for $t \gg \tau$,

$$u_c(t) \approx \frac{1}{1+\tau^2\omega_0^2} \sin(\omega_0 t) - \frac{\tau\omega_0}{1+\tau^2\omega_0^2} \cos(\omega_0 t)$$

This is called the **steady state** of the system. For $t \approx \tau$,we have to take into account the first term which is not negligible, corresponding to **transient state**, i.e. the transition between the off state for $t < 0$ and the steady state for $t \gg \tau$. Finally, note that the steady state is expressed as a function of $\sin(\omega_0 t)$ and $\cos(\omega_0 t)$, namely the sine and cosine with the same period as the input sine $\sin(\omega_0 t)$. We develop this notion in the next part.